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Endogenous Skill Cycles

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Abstratc

This paper explores the ability of a class of one-sector models to generate endogenous skills cycles. Skills cycles are here defined as endogenous fluctuations of the composition of equilibrium allocation of labor services. We consider a one sector economy in which there exist one type of capital stock, and a finite number of different labor services, which are assumed to be heterogeneous along the skill/productivity dimension. We apply the Hopf bifurcation theorem and provide necessary conditions on the model's parameters for having a closed orbit as the economy's stable set. We also develop a numerical example (based on the United States economy) showing how this closed orbit can appear under reasonable parameter values.

JEL Classification: E32, J24.

Keywords: Business fluctuations; Cycles; Skills.

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1 Introduction

This paper explores the ability of a class of one-sector models to generate endogenous skills cycles. Skills cycles are here defined as endogenous fluctuations of the composition of equilibrium allocation of labor services. We consider a one sector economy in which there exist one type of capital stock, and a finite number of different labor services, which are assumed to be heterogeneous along the skill/productivity dimension.[4]

The broader literature discussing dynamical models under indeterminacy is very vast. We only mention some of the papers that are most closely related to our work. Benhabib and Farmer (1994) and Farmer and Guo (1994) discuss indeterminacy and sunspot equilibria in a standard one-sector Real Business Cycle (RBC) model with production externalities (i.e., the model of Baxter and King, 1991). Since this first-generation indeterminate RBC models require implausibly large degrees of externalities to generate indeterminacy (thereby casting doubt on their empirical relevance), subsequent work by Benhabib and Nishimura (1998), Benhabib, Meng and Nishimura (2000), Harrison (2001), Perli (1998), Weder (2003) and Wen (1998), Bennet and Farmer (2000); Hintermaier (2003); Pintus (2007), Loyd-Braga, Nouri, Venditti (2006), among many others, made efforts to reduce the degree of externalities required for inducing local indeterminacy. This line of research discovers that factors such as i) additional sectors of production; ii) durable consumption goods; iii) non-separable utility functions; iv) variable capacity utilization, can each reduce the required degree of increasing returns for local indeterminacy to a figure that is within empirically admissible range. Also the introduction of labor heterogeneity eases the necessity of having an upward sloping labor demand schedules.

The paper shows that under some precisely identified parameters' values detereministic endogenous skill's cycles can arise in an economy with external effects in production. There will be times in which the economy relies on an equilibrium allocation favoring high skilled workers, while other times in which the equilibrium allocation favors low skilled employees. This happens endogenously in our model, and it affects both the aggregate allocation (i.e. level of aggregate labor services employed in equilibrium) and its composition (i.e. how many blu/white collars are employed). In this context we derive analytical condition explaining the topological properties of the model's attractor and the dynamic behavior around it.

The paper is organized as follows. Section 2 presents the theoretical model and its equilibrium; Section 3, then, discusses the topological properties of stationary state and derives conditions for for the endogenous skill cycles. Section 4, next, calibrates the model for the U.S. economy and and provide some numerical examples. Finally, Section 5 concludes.

[4] Notice, however, that what matters is the heterogeneity itself, and it is possible to obtain qualitatively analogous results for different kinds of heterogeneity (i.e. distinguishing between regular and underground labor services, or between labor services spatially separated).

2 The Model

2.1 Firms and households

The paper's model is analogous to that in Busato in Marchetti (2009), thus we consider a continuum of firms and household that differs from the Farmer and Guo (1994) only in the presence of labor heterogeneity. In particular, the $i-$th firm employs aggregate capital stock and M different types of labor services, denoted as n^j ($j = 1, 2, ..., M$) in order to produce an homogenous output $y_{i,t}$ according to production function:

$$y_{i,t} = A_t k_{i,t}^{\alpha_0} \left[\prod_{j=1}^{M} (n_{i,t}^j)^{\alpha_j} \right], \text{ with } \alpha_j > 0; \text{ and } \sum_{j=0}^{M} \alpha_j = 1.$$

The quantity A_t (defined below) represents an aggregate production externality (as in Romer 1986) :

$$A_t = (K_t^{\alpha_0})^{\omega} \prod_{j=1}^{M} \left[(N_t^j)^{\alpha_j} \right]^{\eta_j}, \qquad \omega \neq \eta_j; \quad \omega, \eta_j > 0,$$

where K_t and the $N_t^{j\prime}s$ are the economy-wide levels of the production inputs. The externality effect acts through the capital stock and the various types of labor services; for example, the quantity $\left[(N_t^j)^{\alpha_j} \right]^{\eta_j}$ denotes the external effect associated to the $j-$th type of labor. Finally, the parameters (ω, η_j, $j = 1, 2, ..., M$) can have different values so to exploit the distinctive characteristics of each production factor.

As firms are all identical, overall level of output for a given level of input utilization is given by:

$$Y_t = A_t \int_i y_{i,t} di = K_t^{\alpha_0(1+\omega)} \left[\prod_{j=1}^{M} (N_t^j)^{(1+\eta_j)\alpha_j} \right]. \tag{1}$$

where each individual firm takes K, $N^1, ... , N^M$ as given. As markets are competitive, and returns to scale faced by each firm in production are constant, i.e. $\alpha_0 = 1 - \sum_{j=1}^{m} \alpha_j$, firm's behavior is described by the $M + 1$ first order conditions for the (expected) profit maximization are equal to:

$$\begin{aligned} k_{i,t} &: \alpha_0 \frac{y_{i,t}}{k_{i,t}} = r_t \\ n_{i,t}^1 &: \alpha_1 \frac{y_{i,t}}{n_{i,t}^1} = w_t^1 \\ &\vdots \\ n_{i,t}^M &: \alpha_M \frac{y_{i,t}}{n_{i,t}^M} = w_t^M, \end{aligned} \tag{2}$$

where r_t is the real rate of return on capital and the w_t^j are the real wage rates for each type of labor. All labor services are employed in equilibrium, due to the Cobb-Douglas production structure.

As for the households (symmetrical and indexed with super-script i), each of them consumes $c_{i,t}$ unit of the final good and supplies $j = 1, 2, ..., M$ different types of labor $n_{i,t}^j$; we assume that its preferences are represented by the following separable utility function:

$$\mathcal{V}_{i,t}\left(c_{i,t}, n_{i,t}^1, \cdots, n_{i,t}^M\right) = \log c_t^i - Dn_{i,t} - \sum_{j=1}^{M} \frac{B_j}{1+\psi_j}\left(n_{i,t}^j\right)^{1+\psi_j}.$$

Here we take a cue from Cho and Rogerson's (1988) and Cho and Cooley's (1998) family labor supply model. They distinguish labor supply with regard to an intensive (hours worked), and an extensive margin (employment margin). In our model we reinterpret and generalize these dimensions as representing worker's labor supply in the different segments of the labor market. In particular, household preferences are structured in the following way. Total labor $n_{i,t} = \sum_{j=1}^{M} n_{i,t}^j$ generates an overall disutility of work equal to $Dn_{i,t}$, $D > 0$.[5] In addition each type of labor determines an idiosyncratic disutility $\left[B_j/\left(1+\psi_j\right)\right]\left(n_{i,t}^j\right)^{1+\psi_j}$ with $B_j, \psi_j > 0$, which captures the labor heterogeneity (or labor market segmentation) and are proxies for the labor-specific effort exerted by each household. A possible economic interpretation is to envisage labor heterogeneity as stemming from an un-modeled human capital stock and/or skills. In this case, more productive labor types should display a high marginal productivity (in the steady state equilibrium), matched with a high value for the (steady state) marginal disutility of labor. Then, more skilled labor should be characterized by a relatively high value of B_j.[6] Now, this formulation is not addressing a fully fledged "heterogeneity problem", in particular with respect to consumption choice, but it is looking at a parsimonious model capable of capturing the labor heterogeneity issue.

Next, the household's feasibility constraint ensures that the sum of consumption $c_{i,t}$ and investment $i_{i,t}$ does not exceed consumers' income,

$$c_{i,t} + i_{i,t} = r_t k_{i,t} + \sum_{j=1}^{M} w_t^j n_{i,t}^j,$$

and capital stock is accumulated according to a customary state equation, i.e.: $k_{i,t+1} = (1-\delta)k_{i,t} + i_{i,t}$,where $0 < \delta < 1$ denotes a quarterly capital stock depreciation rate.

Imposing, then, a constant subjective discount rate $0 < \beta < 1$, and defining $\mu_{i,t}$ as the costate variable, we form the Lagrangian of the household's control problem:

$$\mathcal{L}_0^h = E_0 \sum_{t=0}^{\infty} \beta^t \mathcal{V}_{i,t} + E_0 \sum_{t=0}^{\infty} \mu_{i,t}\left(r_t k_{i,t} + \sum_{j=1}^{M} w_t^j n_{i,t}^j - c_{i,t} - i_{i,t}\right).$$

[5] We assume that the disutility coming from aggregate labor is linear in its argument in order to simplify the already complicated algebra.

[6] This is supported by the numerical parameterization chosen for the steady state values of the model. In the numerical example of section 4 (where $M = 2$ and in the simplified case with $D = 0$).

Household's optimal choice is characterized by the following necessary and sufficient conditions:

$$
\begin{aligned}
c_{i,t} &: \quad \beta^t c_{i,t}^{-1} = \mu_{i,t} \\
n_{i,t}^1 &: \quad \beta^t D + \beta^t B_1 \left(n_{i,t}^1\right)^{\psi_1} = \mu_{i,t} w_t^1 \\
&\quad\ \vdots \\
n_{i,t}^M &: \quad \beta^t D + \beta^t B_M \left(n_{i,t}^M\right)^{\psi_M} = \mu_{i,t} w_t^M \\
k_{i,t+1} &: \quad E_t \left\{ \mu_{i,t+1} \left[(1-\delta) + r_{t+1} \right] \right\} = \mu_{i,t} \\
&\quad\ \lim_{t\to\infty} E_0 \mu_{i,t} k_{i,t} = 0
\end{aligned}
\tag{3}
$$

The model collapses to the standard one sector scheme with aggregate increasing returns to scale (e.g. Farmer and Guo [10]) setting $M = 1$ and $\omega = \eta_1 = \eta$ into the previous equilibrium conditions.

2.2 Symmetric perfect foresight equilibrium

A perfect foresight equilibrium is here defined as a sequence of prices $\left\{w_t^1, \cdots, w_t^M, r_t\right\}_{t=0}^{\infty}$ and a sequence of quantities $\left\{N_t^1, \cdots, N_t^M, K_{t+1}, C_t, \right\}_{t=0}^{\infty}$ such that: i) firms and households solve their optimization problems; ii) the resource constraints are satisfied; iii) all markets clear; iv) agents form correct expectations about all relevant future periods given the initial capital stock K_0. As agents are symmetric, aggregate consistency requires that $y_{i,t} = Y_t$, $k_{i,t} = K_t$, $n_{i,t}^j = N_t^j$, $c_t = C_t$, where capital letters denote aggregate equilibrium quantities.

As a result, the equations characterizing the equilibrium are given by:[7]

$$
\begin{aligned}
D + B_1 \left(N_t^1\right)^{\psi_1} &= (C_t)^{-1} \alpha_1 \frac{Y_t}{N_t^1} \\
&\ \ \vdots \\
D + B_M \left(N_t^M\right)^{\psi_M} &= (C_t)^{-1} \alpha_M \frac{Y_t}{N_t^M} \\
(C_{t+1})^{-1} \left((1-\delta) + \alpha_0 \frac{Y_{t+1}}{K_{t+1}} \right) \beta &= (C_t)^{-1} \\
K_t^{\alpha_0(1+\omega)} \prod_{j=1}^{M} \left(N_t^j\right)^{\alpha_j(1+\eta_j)} + (1-\delta) K_t - C_t &= K_{t+1} \\
\lim_{T\to\infty} (C_T)^{-1} K_T &= 0.
\end{aligned}
$$

From the above equations it is possible to derive the steady state, while showing (by a constructive argument) its existence and uniqueness.

[7] The aggregate resource constraint holds: $C_t + I_t = Y_t$.

Proposition 1 *There exists a unique stationary vector of equilibrium capital stock $K^\bigstar > 0$, consumption $C^\bigstar > 0$, and labor services $N^{1\bigstar}, ..., N^{M\bigstar}$ all positive satisfying:*

$$
\begin{aligned}
K^\bigstar &= \left[\frac{1-\beta(1-\delta)}{\alpha_0\beta}\right]^{\frac{1}{\alpha_0(1+\omega)-1}} \left[\left(N^{1\bigstar}\right)^{\frac{\alpha_1(1+\eta_1)}{1-\alpha_0(1+\omega)}} \cdots \left(N^{M\bigstar}\right)^{\frac{\alpha_M(1+\eta_M)}{1-\alpha_0(1+\omega)}}\right] \\
C^\bigstar &= \Xi\left[\prod_{j=1}^{m}\left(N_j^\bigstar\right)^{\alpha_j(1+\eta_j)}\right]^{\frac{1}{1-\alpha_0(1+\omega)}} \\
\left(N^{1\bigstar}\right)^{\psi_{1+1}} &= \frac{\alpha_1}{B_1\Xi}\left[\frac{1-\beta(1-\delta)}{\alpha_0\beta}\right]^{\frac{\alpha_0(1+\omega)}{\alpha_0(1+\omega)-1}} - \frac{D}{B_1}N^{1\bigstar} \\
&\vdots \\
\left(N^{M\bigstar}\right)^{\psi_M+1} &= \frac{\alpha_M}{B_M\Xi}\left[\frac{1-\beta(1-\delta)}{\alpha_0\beta}\right]^{\frac{\alpha_0(1+\omega)}{\alpha_0(1+\omega)-1}} - \frac{D}{B_M}N^{M\bigstar},
\end{aligned}
$$

where $\Xi = \left\{\left[\frac{1-\beta(1-\delta)}{\alpha_0\beta}\right]^{\alpha_0(1+\omega)} - \delta\left[\frac{1-\beta(1-\delta)}{\alpha_0\beta}\right]\right\}^{\frac{1}{\alpha_0(1+\omega)-1}}$ *is a positive quantity defined as a function of the model's parameters.*

Proof. The stationary value for r can be directly calculated from the Euler equation: $r^\bigstar = \frac{1}{\beta} - (1-\delta) > 0$. This value can be substituted into the market demand for capital $r^\bigstar = MPK$ (i.e. the marginal productivity of capital stock), and the resulting equation can be solved w.r.t. K:

$$K = \left[\frac{1-\beta(1-\delta)}{\alpha_0\beta}\right]^{\frac{1}{\alpha_0(1+\omega)-1}} \left[\prod_{j=1}^{M}\left(N^j\right)^{\alpha_j(1+\eta_j)}\right]^{\frac{1}{1-\alpha_0(1+\omega)}} \tag{4}$$

The value of K form equation (4) can be substituted into the resource constraint $C = K^{\alpha_0(1+\omega)}\left[\prod_{j=1}^{M}\left(N^j\right)^{\alpha_j(1+\eta_j)}\right] - \delta K$ yielding:

$$
\begin{aligned}
C &= \left[\frac{r^\bigstar}{\alpha_0}\right]^{\frac{\alpha_0(1+\omega)}{\alpha_0(1+\omega)-1}} \left[\prod_{j=1}^{M}\left(N^j\right)^{\alpha_j(1+\eta_j)}\right]^{1+\frac{\alpha_0(1+\omega)}{1-\alpha_0(1+\omega)}} \\
&\quad -\delta\left[\frac{r^\bigstar}{\alpha_0}\right]^{\frac{1}{\alpha_0(1+\omega)-1}} \left[\prod_{j=1}^{M}\left(N^j\right)^{\alpha_j(1+\eta_j)}\right]^{\frac{1}{1-\alpha_0(1+\omega)}}
\end{aligned}
$$

or also:

$$C = \Xi\left[\prod_{j=1}^{M}\left(N^j\right)^{\alpha_j(1+\eta_j)}\right]^{\frac{1}{1-\alpha_0(1+\omega)}}, \tag{5}$$

where: $\left\{\left[\frac{1-\beta(1-\delta)}{\alpha_0\beta}\right]^{\alpha_0(1+\omega)} - \delta\left[\frac{1-\beta(1-\delta)}{\alpha_0\beta}\right]\right\}^{\frac{1}{\alpha_0(1+\omega)-1}} = \Xi > 0$. Note that for our two parameterizations (shown in section 4) the value of Ξ equals 1.819 and 1.687 consistently with our requirement. The value of K from equation (4) can now be substituted into the equilibrium equations for the labor markets, together with the value of C from (5), so to obtain:

$$\begin{aligned} D + B_1 \left(N^1\right)^{\psi_1} &= \frac{\alpha_1}{\Xi}\left[\frac{r^\bigstar}{\alpha_0}\right]^{\frac{\alpha_0(1+\omega)}{\alpha_0(1+\omega)-1}} \left(N^1\right)^{-1} \\ &\vdots \\ D + B_M \left(N^M\right)^{\psi_M} &= \frac{\alpha_M}{\Xi}\left[\frac{r^\bigstar}{\alpha_0}\right]^{\frac{\alpha_0(1+\omega)}{\alpha_0(1+\omega)-1}} \left(N^M\right)^{-1}, \end{aligned}$$

which can be written in this way:

$$\begin{aligned} \left(N^1\right)^{\psi_1+1} &= \Theta_1 - \frac{D}{B_1}N^1 \\ &\vdots \\ \left(N^M\right)^{\psi_M+1} &= \Theta_M - \frac{D}{B_M}N^M, \end{aligned} \tag{6}$$

with $\Theta_j = \frac{\alpha_j}{B_j\Xi}\left[\frac{r^\bigstar}{\alpha_0}\right]^{\frac{\alpha_0(1+\omega)}{\alpha_0(1+\omega)-1}} > 0$. Each of the (6) can be thought of as an equality between two functions of the same (and unique) variable N^j: one is a straight line $\Theta_j - (D/B_j)\,N^j$ with positive intercept Θ_j and negative slope $-(D/B_j)$; the other one is a monotonically increasing function $\left(N^j\right)^{\psi_j+1}$ crossing the origin of the axis. Thus each of the (6) determines a positive, single and unique equilibrium value $N^{j\bigstar}$. The vector of stationary values $\left(N^{1\bigstar}, \cdots, N^{M\bigstar}\right)$ computed from equations (6) can be substituted into equations (4) and (5) so to determine the unique stationary values $K^\bigstar$ and $C^\bigstar$. ■

3 Topological properties and endogenous cycles

To solve the model, we log-linearize the economy-wide equilibrium conditions around the steady state derived in Proposition 1. Denoting with S_t as the vector $(K_t; C_t)$, the model can be reduced to the following system of linear difference equations (where hat-variables denote percentage deviations from their steady state values):

$$\widehat{S}_{t+1} = \mathbf{F}\widehat{S}_t, \tag{7}$$

where $\mathbf{F}$ is a coefficient matrix. Consider the equilibrium equations from section 2.2 of the M labor markets: $D + B_j\left(N_t^j\right)^{\psi_j} = (C_t)^{-1}\alpha_j\frac{Y_t}{N_t^j}$, $j = 1, ..., M$. The first equation (where $j = 1$) can be rewritten in this way: $C_t = \alpha_1\frac{Y_t}{B_1\left(N_t^1\right)^{1+\psi_1}}$, and by substituting this expression into the remaining $M-1$ market equilibrium conditions we obtain:

$$\begin{aligned}
\frac{DN_t^2 + B_2\left(N_t^2\right)^{1+\psi_2}}{\alpha_2} &= \frac{DN_t^1 + B_1\left(N_t^1\right)^{1+\psi_1}}{\alpha_1} \\
&\vdots \\
\frac{DN_t^M + B_M\left(N_t^M\right)^{1+\psi_M}}{\alpha_M} &= \frac{DN_t^1 + B_1\left(N_t^1\right)^{1+\psi_1}}{\alpha_1}
\end{aligned}$$

These equations can be linearized around a neighborhood of the steady state, so to obtain[8]**:**

$$\begin{aligned}
\widehat{N}_t^2 &= \left(\frac{1+\psi_1 S_1}{1+\psi_2 S_2}\right)\widehat{N}_t^1 \\
&\vdots \\
\widehat{N}_t^M &= \left(\frac{1+\psi_1 S_1}{1+\psi_M S_M}\right)\widehat{N}_t^1,
\end{aligned} \tag{8}$$

where $S_j = \left(\frac{D}{B_j (N^{j\bigstar})^{\psi j}} + 1\right)^{-1}$ for $j \geq 2$. The further step is to linearize the equilibrium condition $C_t = \alpha_1 \frac{Y_t}{B_1\left(N_t^1\right)^{1+\psi_1}}$, and to combine with equations (8); next, the resulting equation can be solved with respect to $\widehat{N}_t^1$:

$$\widehat{N}_t^1 = \left[(1+\psi_1 S_1)(\Phi-1)\right]^{-1}\widehat{C}_t - \left[\frac{(1+\omega)\alpha_0}{(1+\psi_1 S_1)(\Phi-1)}\right]\widehat{K}_t, \tag{9}$$

where $\Phi = \sum_{j=1}^{M} \frac{(1+\eta_j)\alpha_j}{1+\psi_j S_j}$. It is now possible to construct the 2×2 dynamic system at the heart of the economic model. The $\widehat{N}_t^j$ values from (8) and (9) can be combined with the demand for capital $\widehat{r}_{t+1} = \left[(1+\omega)\alpha_0 - 1\right]\widehat{K}_{t+1} + \sum_{j=1}^{M}\left[(1+\eta_j)\alpha_j\right]\widehat{N}_{t+1}^j$, yielding:

$$\widehat{r}_{t+1} = \left[(1+\omega)\alpha_0\left(1+\varphi(\mathbf{P})\right) - 1\right]\widehat{K}_{t+1} - \left[\varphi(\mathbf{P})\right]\widehat{C}_t \tag{10}$$

where, indicating the set of our model parameters by $\mathbf{P}$, we define $\varphi(\mathbf{P}) = \frac{\Phi}{1-\Phi}$ as a continuous mapping such that: $\varphi(\mathbf{P}) : \mathbf{P} \mapsto \Re$. Now, recalling the stationary value for the interest rate: $r^\bigstar = \frac{1}{\beta} - 1 + \delta$, the linearized version of the Euler equation $E_t(C_{t+1})^{-1}\beta\left((1-\delta) + r_{t+1}\right) = (C_t)^{-1}$ is given by: $\widehat{C}_{t+1} - \left[1-\beta(1-\delta)\right]\widehat{r}_{t+1} = \widehat{C}_t$. Equation (10) can be substituted in turn into the linearized Euler equation, obtaining:

$$\begin{aligned}
\widehat{C}_t = {} & \left\{1 + \left[1-\beta(1-\delta)\right]\varphi(\mathbf{P})\right\}\widehat{C}_{t+1} \\
& - \left\{\left[1-\beta(1-\delta)\right]\left[(1+\omega)\alpha_0\left(1+\varphi(\mathbf{P})\right) - 1\right]\right\}\widehat{K}_{t+1}.
\end{aligned} \tag{11}$$

From the linearization of the budget constraint, we obtain the second dynamic equation:

[8] We adopt the following definition: for each variable x_t, $\widehat{x}_t = \frac{x_t - x^\bigstar}{x^\bigstar}$, where $x^\bigstar$ is the steady state value of x_t.

$$\left[\frac{K^\star}{Y^\star}\right]\widehat{K}_{t+1} = \widehat{Y}_t - \left[\frac{C^\star}{Y^\star}\right]\widehat{C}_t + \left[(1-\delta)\frac{K^\star}{Y^\star}\right]\widehat{K}_t,$$

where starred variables indicates the steady state values. By using the linearized production function $\widehat{Y}_t = (1+\omega)\alpha_0\widehat{K}_t + \sum_{j=1}^{M}\left[(1+\eta_j)\alpha_j\right]\widehat{N}_t^j$ and equations (9), total output deviation is equal to: $\widehat{Y}_t = (1+\omega)\alpha_0\left(1+\varphi(\mathbf{P})\right)\widehat{K}_t - \left(\varphi(\mathbf{P})\right)\widehat{C}_t$. Thus the budget constraint turns out to be:

$$[s_I]\,\widehat{K}_{t+1} = \left[\delta\alpha_0(1+\omega)\left(1+\varphi(\mathbf{P})\right) + (1-\delta)s_I\right]\widehat{K}_t - \left[\delta\varphi(\mathbf{P}) + \delta s_C\right]\widehat{C}_t \tag{12}$$

where $s_C = \frac{C^\star}{Y^\star}$, $\frac{s_I}{\delta} = \frac{K^\star}{Y^\star}$ and $s_C + s_I = 1$.

Equations (11) and (12) define the equilibrium dynamic system of the model:

$$\begin{bmatrix} -J_1 & J_2 \\ s_I & 0 \end{bmatrix}\begin{bmatrix} \widehat{K}_{t+1} \\ \widehat{C}_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ J_3 & -J_4 \end{bmatrix}\begin{bmatrix} \widehat{K}_t \\ \widehat{C}_t, \end{bmatrix}$$

where the J's are defined below:

$$\begin{aligned} J_1 &= \left[1-\beta(1-\delta)\right]\left\{\alpha_0(1+\omega)\left[1+\varphi(\mathbf{P})\right]-1\right\}; \\ J_2 &= 1+\left[1-\beta(1-\delta)\right]\varphi(\mathbf{P}); \\ J_3 &= \delta\alpha_0(1+\omega)\left(1+\varphi(\mathbf{P})\right)+(1-\delta)s_I; \\ J_4 &= \delta\varphi(\mathbf{P})+\delta s_C. \end{aligned}$$

Now, if $J_2 \neq 0$, i.e. if $\frac{\Phi}{1-\Phi} \neq \frac{-1}{1-\beta(1-\delta)}$, the system can be put in ordinary form:

$$\begin{bmatrix} \widehat{K}_{t+1} \\ \widehat{C}_{t+1} \end{bmatrix} = \mathbf{F}\begin{bmatrix} \widehat{K}_t \\ \widehat{C}_t \end{bmatrix} \quad \text{where:} \quad \mathbf{F} = \begin{bmatrix} \frac{1}{s_I}J_3 & -\frac{1}{s_I}J_4 \\ \frac{J_1}{J_2 s_I}J_3 & \frac{1}{J_2} - \frac{J_1}{J_2 s_I}J_4 \end{bmatrix} \tag{13}$$

As the variable K_t is predetermined, the model's topological properties will depend on the moduli of the eigenvalues $\mathbf{F}$; for example, system (13) displays indeterminacy of the equilibrium path if both the eigenvalues lie inside the unit circle. Necessary and sufficient conditions for local indeterminacy of the equilibrium path are derived in Busato and Marchetti (2009). Indeterminacy conditions suggest (intuitively speaking) that linearized labor demand functions should react relatively more to changes in capital stock rather than changes in labor services, and that, *ceteris paribus,* labor supply functions should be sufficiently elastic.[9] In other words, each labor demand schedule should display a large enough response to variation in capital stock for expectation to be self-fulfilled.

In general, the model's properties depends upon the values of the externality parameters η_j, and there can also be stable sets qualitatively different from a point. In particular it is possible to show that for adequate parameters values, the stable set of system (13) is

[9]Technically speaking, for the generic inverse demand function of labor of type i, the term $\frac{\partial\widehat{w}^i}{\partial\widehat{K}}^{d}$ should be larger than $\sum_{j=1}^{M}\frac{\left(\partial\widehat{w}^i/\partial\widehat{N}^j\right)^d}{1+\left(\partial\widehat{w}^i/\partial\widehat{N}^j\right)^s}$, which is also reduced by quantities $\frac{s_I}{\delta}$ and $(1-\delta)(1-\beta)$.

a closed orbit in this case the economy could be characterized by endogenous cycles. For establishing the possible presence of such cycles, it is necessary to know if (13) undergoes to a Hopf bifurcation under perturbation of selected deep parameters $(\alpha_j, \beta, \psi_j, \delta, \eta_j, \omega, B_j, D)$.

In applying the bifurcation analysis to system (13), our first step will be to introduce a simplification in our former assumptions: we will set parameter D to 0, so that the term Φ will be equal to $\Phi = \sum_{j=1}^{M} \frac{(1+\eta_j)\alpha_j}{1+\psi_j}$ (i.e. $S_j = 1$ for all the j's) and will depend only on the parameters $(\alpha_j, \eta_j, \psi_j)$. This simplification not only eases the choice of the bifurcation parameter, but is also necessary for removing the dependence of Φ on the steady state values of the labour inputs.[10]

We can state the following:

Theorem 2 *Assume $D = 0$. If there exist a string of parameters values $(\bar{\eta}_{j=1,2\cdots n})$ such that: **i)** eigenvalues of* $\mathbf{F}$ *are complex conjugate; **ii)** $\alpha_0 (1+\omega) = \frac{s_I}{\delta} \left[1 + \bar{\Phi}(1-\beta)(1-\delta)\right]$, $\bar{\Phi} = \sum_{j=1}^{M} \frac{(1+\bar{\eta}_j)\alpha_j}{1+\psi_j}$ given that: $\omega > 0$; **iii)** the trace of* $\mathbf{F}$ *computed in $\Phi = \bar{\Phi}$, i.e.:$Tr(\mathbf{F})|_{\bar{\Phi}} = Tr_{\bar{\Phi}}$, satisfies the following conditions: $Tr_{\bar{\Phi}} \neq \pm 1, \pm\sqrt{2}, \pm 2$; then there is an invariant closed curve bifurcating from $\bar{\Phi}$.*

Preliminaries. The characteristic polynomial of $\mathbf{F}$ is given by: $\lambda^2 - Tr(\mathbf{F})\lambda + Det(\mathbf{F})$ where λ are the eigenvalues of $\mathbf{F}$ while Tr and Det are, respectively, its trace and determinant defined below:

$$Det(\mathbf{F}) = \frac{\delta\alpha_0(1+\omega)(1+\varphi(\mathbf{P}))+(1-\delta)s_I}{s_I[1+[1-\beta(1-\delta)]\varphi(\mathbf{P})]}$$
$$Tr(\mathbf{F}) = \frac{s_I+\{\delta\alpha_0(1+\omega)(1+\varphi(\mathbf{P}))+(1-\delta)s_I\}\{1+[1-\beta(1-\delta)]\varphi(\mathbf{P})\}-\{[1-\beta(1-\delta)]\{\alpha_0(1+\omega)[1+\varphi(\mathbf{P})]-1\}\}[\delta\varphi(\mathbf{P})+\delta s_C]}{s_I\{1+[1-\beta(1-\delta)]\varphi(\mathbf{P})\}}$$

The planar system (13) can be studied by using the standard methods of Azariadis (1993) or Grandmont et. al. (1998).

Proof. The proof stems from the application of the Hopf bifurcation theorem (existence part) to system (13). Hopf theorem can be stated in the following way.[11] Let the mappig $\mathbf{x}_{t+1} = F(\mathbf{x}_t, \xi)$, $F \in \Re^2 \to \Re^2$, $\xi \in \Re$ (ξ is a parameter) have a smooth family of fixed points $\mathbf{x}^*(\xi)$ at which the linear approximation $\mathbf{x}_{t+1} = \mathbf{F}(\xi)\mathbf{x}_t$, $\mathbf{F} = \frac{\partial F}{\partial \mathbf{x}}$ has complex conjugate eigenvalues λ. If there exist a ξ_0 such that: **a)** $mod\lambda(\xi_0) = 1$, but $\lambda(\xi_0)^n \neq \pm 1$, for $n = 1, 2, 3, 4$; **b)** $\frac{d(mod\lambda(\xi_0))}{d\xi} > 0$, then there is an invariant closed curve bifurcating from $\xi = \xi_0$.

We show that there exist a non-empty set of values for the model's parameters such that each of the three above-mentioned conditions holds when applied to $\mathbf{F}$ of system (13). The proofs of each of conditions in **a)-b)** are the object of three distinct claims.

Claim 3 *If $\alpha_0 (1+\omega) = \frac{s_I}{\delta} \left[1 + \bar{\Phi}(1-\beta)(1-\delta)\right]$ and $-2 < Tr_{\bar{\Phi}} < 2$, then matrix* $\mathbf{F}$ *of system (13) has two complex conjugated eigenvalues with modulus equal to 1.*

---

[10] Note that when $D > 0$ and $\Phi = \sum_{j=1}^{M} \frac{(1+\eta_j)\alpha_j}{1+\psi_j S_j}$, the terms $S_j = \left[\frac{D}{B_j\left(N^{j\bigstar}\right)^{\psi_j}} + 1\right]^{-1}$ will be different from 1 and will depend on the $N^{j\bigstar}$.

[11] Which is taken from Iooss (1979) and Guckenheimer and Holmes (1983). See also Lorenz (1993), p. 115.

Proof. Eigenvalues of $\mathbf{F}$ are equal to: $\lambda_{1,2} = -\frac{Tr(\mathbf{F})}{2} \pm \sqrt[2]{\frac{Tr(\mathbf{F})^2}{4} - Det(\mathbf{F})}$, and in general λ can be written in the following way: $\lambda_{1,2} = h_1 \pm ih_2$, where $h_1 = -\frac{Tr(\mathbf{F})}{2}$, $h_2 = \frac{1}{2}\sqrt[2]{4Det(\mathbf{F}) - Tr(\mathbf{F})^2}$. The modulus of λ is given by the expression: $mod(\lambda_{1,2}) = \sqrt{h_1^2 + h_2^2} = \sqrt{Det(\mathbf{F})}$, so that $mod(\lambda_{1,2}) = 1$ iff $Det(\mathbf{F}) = 1$. We choose $\varphi = \Phi/(1-\Phi)$ as the bifurcation parameter[12] ξ, and check the restrictions on φ for having $Det(\mathbf{F}) = 1$. By using the expression of $Det(F)$ previously defined, we have the following restriction:

$$mod(\lambda_{1,2}) = 1 \Longleftrightarrow \bar{\varphi} = \frac{\alpha_0(1+\omega) - s_I}{\alpha_0\beta - \alpha_0(1+\omega)}$$

We need to show that when $\varphi = \bar{\varphi}$, the two roots are complex conjugated. This is the case when $(Tr_{\bar{\Phi}})^2 - 4 < 0$, i.e. when $-2 < Tr_{\bar{\Phi}} < 2$. Finally, straightforward algebra shows that: $\alpha_0(1+\omega) = \frac{s_I}{\delta}\left[1 + \bar{\Phi}(1-\beta)(1-\delta)\right] \Longleftrightarrow \bar{\varphi} = \frac{\alpha_0(1+\omega)-s_I}{\alpha_0\beta - \alpha_0(1+\omega)}$. ■

Claim 4 *When $Tr_{\bar{\Phi}} \neq \pm 1, \pm\sqrt{2}, \pm 2$, the roots of $\mathbf{F}$ are imaginary and satisfy: $\lambda(\bar{\varphi})^n \neq \pm 1$, for $n = 1, 2, 3, 4$.*

Proof. Note that given any complex number $\lambda = h_1 + ih_2$, the following formulas holds:

$$\begin{aligned}
\lambda &= h_1 + ih_2 \\
\lambda^2 &= h_1^2 - h_2^2 + i(2h_1h_2) \\
\lambda^3 &= h_1^3 - 3h_1h_2^2 + i(3h_1^2h_2 - h_2^3) \\
\lambda^4 &= h_1^4 - 6h_1^2h_2^2 + 4h_2^4 + i(4h_1^3h_2 - 4h_1h_2^3)
\end{aligned}$$

Thus it is: $Tr_{\bar{\Phi}} = \pm 2 \Longleftrightarrow (\lambda(\bar{\varphi}) \text{ is real}) \wedge (\lambda(\bar{\varphi})^2 = 1)$ – in fact $\lambda(\bar{\varphi})$ is real iff $\lambda(\bar{\varphi})^2 = 1$. As a result, the first two conditions of the claim, $(\lambda(\bar{\varphi}))^{j=1,2} \neq \pm 1$, hold when $Tr_{\bar{\Phi}} \neq \pm 2$. We now pass to λ^3; the imaginary part of λ^3 is equal to $Im(\lambda^3) = 3h_1^2h_2 - h_2^3 = h_2(3h_1^2 - h_2^2)$. For our eigenvalues it is: $h_2|_{\bar{\varphi}} = \sqrt{1 - \frac{(Tr_{\bar{\Phi}})^2}{4}}$ and $h_1|_{\bar{\varphi}} = -\frac{Tr_{\bar{\Phi}}}{2}$, so that the imaginary part is equal to:

$$Im(\lambda(\bar{\varphi})^3) = \left(\sqrt{1 - Tr_{\bar{\Phi}}^2/4}\right)\left(Tr_{\bar{\Phi}}^2 - 1\right)$$

Then $Im(\lambda(\bar{\varphi})^3)$ is different form 0 when $Tr_{\bar{\Phi}} \neq \pm 1$ and $Tr_{\bar{\Phi}} \neq \pm 2$. Let us consider λ^4, whose imaginary part is: $Im(\lambda^4) = 4h_1h_2(h_1^2 - h_2^2)$. By using our expression for $h_2|_{\bar{\varphi}}$ and $h_1|_{\bar{\varphi}}$, we have:

$$Im(\lambda(\bar{\varphi})^4) = -2Tr_{\bar{\Phi}}\left(\sqrt{1 - Tr_{\bar{\Phi}}^2/4}\right)\left(\frac{Tr_{\bar{\Phi}}^2}{2} - 1\right)$$

As before, for having $Im(\lambda(\bar{\varphi})^4) \neq 0$ it must be $Tr_{\bar{\Phi}} \neq \pm 2$, but also the inequality $Tr_{\bar{\Phi}} \neq \pm\sqrt{2}$ must be satisfied. Finally, for completeness, recall that the analytical expression of

[12] Because Φ depends only on the $(\alpha_j, \psi_j, \eta_j,\ j = 1, ..., M)$ fundamental parameters.

$Tr_{\bar{\Phi}}$ is:

$$
\begin{aligned}
Tr_{\bar{\Phi}} \quad = \quad & \frac{\delta \left\{ \alpha_0 \beta \left[\alpha_0 (1+\omega) \delta + 1 - \delta \right] - \beta (1-\delta) \left[1 + \alpha_0 (1+\omega) \right] + 1 \right\} \left(\frac{\bar{\Phi}}{1-\bar{\Phi}} \right)}{s_I \left[1 + \left[1 - \beta (1-\delta) \right] \frac{\bar{\Phi}}{1-\bar{\Phi}} \right]} \\
& + \frac{\delta \left[\beta (1-\delta) + \alpha_0 \beta \delta \right] \left[\alpha_0 (1+\omega) - 1 \right] + (2-\delta) s_I + \delta}{s_I \left[1 + \left[1 - \beta (1-\delta) \right] \frac{\bar{\Phi}}{1-\bar{\Phi}} \right]}
\end{aligned}
$$

and as $\frac{\bar{\Phi}}{1-\bar{\Phi}} = \frac{\alpha_0(1+\omega)-s_I}{\alpha_0\beta-\alpha_0(1+\omega)}$, $Tr_{\bar{\Phi}}$ depends only on the $\alpha_0, \beta, \delta, \omega$ parameters. ∎

Claim 5 $\omega > 0 \Longrightarrow \frac{d(mod\lambda(\bar{\Phi}))}{d\Phi} > 0$.

Proof. Recall that: $mod(\lambda_{1,2}) = \sqrt{Det(\mathbf{F})}$., thus we can compute:

$$
\left. \frac{d(mod\lambda(\bar{\Phi})}{d\Phi} \right|_{\bar{\Phi}} = \frac{(1-\delta)}{2\left(1-\bar{\Phi}\right)^2 \sqrt{Det(\mathbf{F})|_{\bar{\Phi}}}} \left\{ \frac{\delta\alpha_0(1+\omega)\beta - \left[1-\beta(1-\delta)\right] s_I}{s_I \left\{1 + \left[1-\beta(1-\delta)\right]\bar{\Phi}\right\}^2} \right\}
$$

Now, insert in the above expression: $\sqrt{Det(\mathbf{F})|_{\bar{\Phi}}} = 1$ and $s_I = \frac{\alpha_0\beta\delta}{1-\beta(1-\delta)}$, so to obtain:

$$
\left. \frac{d(mod\lambda(\bar{\Phi})}{d\Phi} \right|_{\bar{\Phi}} = \frac{\alpha_0\beta\delta(1-\delta)\omega}{2s_I\left(1-\bar{\Phi}\right)^2 \left\{1 + \left[1-\beta(1-\delta)\right]\bar{\Phi}\right\}^2} > 0.
$$

∎ ∎

Thus, when the model's parameter satisfy the three conditions **i)-iii)** of Theorem 2 the economy can have a closed orbit as stable set. To show that this actually occurs for meaningful parameters' values, we examine some numerical examples.

4 Parametrization and numerical examples

The model is then parameterized for the United States economy. We consider two types of labor services, skilled and unskilled, following the OECD definition (more details below). Then matrix $\mathbf{F}$ depends on a set of twelve parameters. Five pertain to household preferences, $(\psi_1, \psi_2, B_1, B_2, \beta)$, and seven to technology (the private capital share α_0, the unskilled and skilled labor shares α_1, α_2, the corresponding externality coefficients ω, η_1, η_2, respectively, and the depreciation rate δ). Three of these parameters are calibrated according to standard estimates for this type of models (see for example Farmer and Guo (1994)): $\beta = 0.984$, $\delta = 0.025$ and $\alpha_0 = 0.23$.

Skilled-unskilled labor have been identified using OECD data for the U.S. economy;[13] according to these data, the average value (for the 1997-2000 period) of the share of total

[13]Data source: OECD (2004), table 4 Labor Force Statistics by educational attainment (for the United States). List of time series: ISCED 0/1 Series Name U17 E0 2032; ISCED 2 Series Name U17 E0 2232; ISCED 3A Series Name U17 E0 2432; ISCED 5A/6 Series Name U17 E0 2B32; ISCED 5B Series Name U17 E0 2C32.

labor force with higher education (ISCED 5A6 - 5B) equals 34.03%, giving rise to a steady state ratio for $\left(\frac{N^1}{N^2}\right)^*$ of 1.94 (starred variables denote calibrated quantities). The parameter B_2 is used for calibrating the ratio between unskilled and skilled workers to that value; by imposing $B_2^* = 0.9$ we obtain a value of 1.97 for the ratio $\left(N^1/N^2\right)^*$. The calibrated parameter $B_1^* = 0.523$ is qualitatively consistent with Imai and Keane (2004). Technology parameters α_1, α_2 are calibrated as follows. Papageorgiou (2001) estimates a production function with skilled and unskilled labor components for the United States economy suggesting that the share of skilled labor α_2^* can be calibrated to 0.36, and the unskilled labor share α_1 equals 0.41.

Given inputs' shares α_j, the key term $\bar{\Phi}$ (i.e. our bifurcation parameter), depends only on the externality parameters η_j and on the labor supply elasticities ψ_j, while also the externality on capital ω remains to be fixed. By choosing appropriate values for these five parameters, it is possible to show how the model displays a closed orbit. Consider the two following parameterizations $\mathbf{P}_{\text{high}}$ and $\mathbf{P}_{\text{low}}$:

	η_1	η_2	ω	ψ_1	ψ_2	RTS degree
$\mathbf{P}_{\text{high}}$	0.535	1.1	0.11	0.09	0.01	1.6407
$\mathbf{P}_{\text{low}}$	0.315	0.4124	0.002	0	0	1.2781

The corresponding matrices $\mathbf{F}$ are then (approximately) equal to:

$$\mathbf{F}\left(\mathbf{P}_{\text{high}}\right) = \begin{bmatrix} 0.8376 & 0.5631 \\ -0.0717 & 1.1457 \end{bmatrix}; \quad \mathbf{F}\left(\mathbf{P}_{\text{low}}\right) = \begin{bmatrix} 0.1067 & 3.7926 \\ -0.2371 & 0.9439 \end{bmatrix}.$$

$$\lambda_{\mathbf{P}_{\text{high}}} = 0.99165 \pm 0.12901i; \qquad \lambda_{\mathbf{P}_{\text{low}}} = 0.5253 \pm 0.8509i$$

thus the eigenvalue for the two parametrization, $\lambda_{\mathbf{P}_{\text{high}}}$ and $\lambda_{\mathbf{P}_{\text{low}}}$ have both absolute value equal to 1. Also the two values of the trace, $Tr\left(\mathbf{P}_{\text{high}}\right)$ and $Tr\left(\mathbf{P}_{\text{low}}\right)$, are consistent with the restrictions mentioned in Theorem 2, as they are respectively equal to: 1.9833 and 1.0506.

For $\mathbf{P}_{\text{high}}$ and $\mathbf{P}_{\text{low}}$ the model possess a closed orbit, and it is interesting to note that the degree of returns to scale (RTS) under $\mathbf{P}_{\text{high}}$ is very close to that of the original model of Farmer and Guo (1994). As is well known, this latter value is implausibly high, but note that the RTS degree under $\mathbf{P}_{\text{low}}$, (1.2781) is relatively close to the value suggested by recent empirical estimates for the U.S. economy (≤ 1.2).

The presence of the bifurcation can also be viewed by tracing the modulus and the imaginary part of the eigenvalues $\lambda_{\mathbf{P}_{\text{high}}}$ and $\lambda_{\mathbf{P}_{\text{low}}}$ when η_2 changes. This is shown in figure 1 below.

For small values of η_2, both $\lambda_{\mathbf{P}_{\text{high}}}$ and $\lambda_{\mathbf{P}_{\text{low}}}$ are real, and the modulus of the eigenvalue is lower than 1. As η_2 increases the largest eigenvalue passes through 1; actually, when η_2 is around a specific value (0.31 for $\mathbf{P}_{\text{high}}$ and 0.4 for $\mathbf{P}_{\text{low}}$) the system undergoes a significant topological change: it can be shown that the largest eigenvalue changes sign form positive infinity to negative infinity (these extreme values are scaled down in figure 1 so to obtain a readable graphic) and the stable set is a sink. Subsequently, when η_2 reaches a threshold

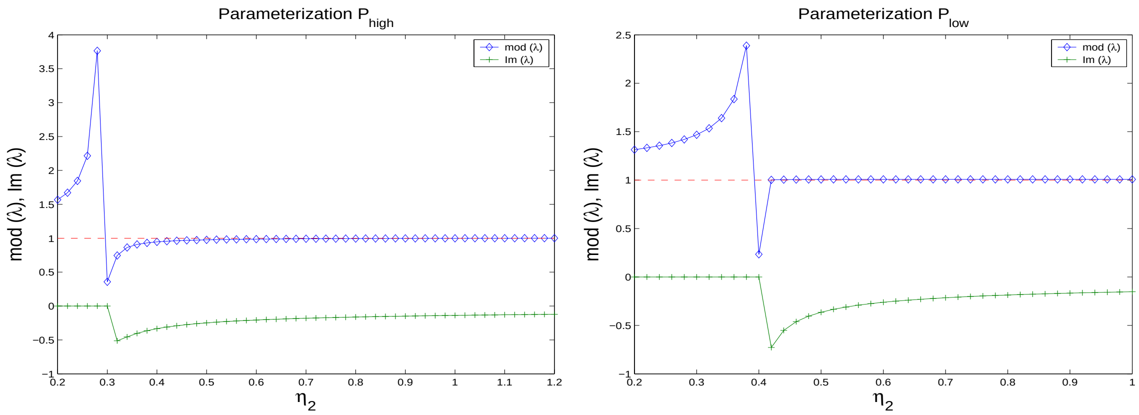

Figure 1: The figure shows the modulus of the eigenvalue $\lambda = -Tr(\mathbf{F})/2 + \sqrt[2]{Tr(\mathbf{F})^2/4 - Det(\mathbf{F})}$ and its imaginary part for different values of η_2, leaving the other parameters as in $\mathbf{P}_{\text{high}}$ (left panel) and $\mathbf{P}_{\text{low}}$ (right panel). Note that in the right panel $mod\,(\lambda)$ is greater than 1 for $\eta_2 > 0.4124$, although the values are small with respect to the vertical axis' unit of measure.

value (approximately 1.1 for $\mathbf{P}_{\text{high}}$ and 0.4124 for $\mathbf{P}_{\text{low}}$), the eigenvalues are still complex conjugated and their modulus equals unity; for these values system (13) posses a closed orbit as stable set.

5 Conclusions and economic intuition

The condition derived in Theorem 2 characterizes the economic mechanism explaining the model's equilibrium dynamics. The theorem derives the condition under which the equilibrium allocations of differently skilled workers endogenously fluctuate along a closed orbit centered on the system's fixed point.

When the economy is positioned in a closed orbit, as, for instance, is the case of two labor types in parameterizations P_{high} and P_{low}, the condition ii) mentioned in Theorem 2 can be written in this way:

$$\epsilon_{i,k}^{d} = \frac{s_I}{\delta}\left[1 + (1-\beta)(1-\delta)\sum_{j=1}^{M}\frac{\epsilon_{j,i}^{d}}{1+\epsilon_j^{s}}\right] \text{ for } i,j = 1,2, \tag{14}$$

where $\epsilon_{i,k}^{d} = \alpha_0(1+\omega)$ is the elasticity of the i-th (j-th) labor demand schedule to capital stock; $\epsilon_j^{d} = (1+\bar{\eta}_j)\,\alpha_j$ denotes the elasticity of the demand for labor of type i to variations of the j-th type of labor (the cross-elasticity i-j); $\epsilon_j^{s} = \psi_j$ is the supply elasticity of the j-th type of labor. Given that the slope of the j-th linearized demand function equals to $(1+\bar{\eta}_j)\,\alpha_j - 1$, this condition suggests that, along the endogenous cycles, each labor demand function should react to changes in capital stock in a very precise way, i.e. in the "right" proportion with respect to changes in labor services.[14]

It is useful to compare this intuition with the case in which the system's attractor is a sink. In this situation, it is possible to show (see: Busato and Marchetti (2009) that the following condition holds:[15]

$$\epsilon_{i,k}^{d} > \frac{s_I}{\delta}\left[1 + (1-\beta)(1-\delta)\sum_{j=1}^{M}\frac{\epsilon_{j,i}^{d}}{1+\epsilon_{j}^{s}}\right] \text{ for } i,j = 1,2. \tag{15}$$

so that in this case the impact coefficient $\epsilon_{i,k}^{d} = \frac{\partial \widehat{w}_t^i}{\partial \widehat{K}_t}$ must be "high enough" with respect to the aggregate impact $\Phi = \sum_{j=1}^{M=2} \frac{\epsilon_{j,\kappa}^{d}}{1+\epsilon_{j}^{s}}$. The mechanism can be described in the following way. Suppose that the economy is initially in its (sink) stationary solution; then a temporary sunspot shock (implying a higher income-production) hits the system. For the agents expectations to be fulfilled, the reactions of the labour demand functions must be such that inequality (15) holds. This imply that a relatively strong (and positive) impact of the capital stock on labor demand ($\epsilon_{i,k}^{d}$) is required in order to offset the initial negative effect due to the cross elasticities of labor. We assume that the η_j are compatible with labor demand functions which are negatively sloped (with respect to the own wage rate)[16], while the labor supply schedules are positively sloped ($\psi_j > 0$). When the expansionary sunspot shock hits the system, the initial reaction can be represented by an upward shift in the labor supply functions (via, e.g., an increase in the expected C, see equations (3)): the agents want to consume more and work less (for a given wage rate). As the labor demand functions are negatively sloped, the (temporary) equilibrium in the labor markets would imply a reduction in the usage of each labor input, and the amplitude of this "negative" effect is given by the cross elasticities $\epsilon_{j,i}^{d}$. This would lead to a recession rather than to an expansion, which was actually the initial "sunspot" conjecture. But if the agents do believe in the sunspot, they will also want to expand the capital stock, and this expansion would, in turn, back-propagate into the labor demand functions via the elasticity $\epsilon_{i,k}^{d}$. If the value of $\epsilon_{i,k}^{d}$ is high enough with respect to that of the $\epsilon_{j,i}^{d}$'s, (as stated in inequality (15)) then the net effect on the labor demand functions will be "positive" (expansionary), and the same functions will shift upward: the equilibrium outcome will be an increase of the usage of the labor inputs (as well as an increase of the wage rates).

But nonetheless, the effect of a sunspot shock is transitory; when the sunspot (in the following period) reverts to its mean value (zero), the agents will update their choices by reducing the inputs' equilibrium levels (as well as production and consumption). The

[14]The linearized demand function for labour type j is given by:

$$\widehat{w}_t^j = [(1+\omega)\alpha_0]\,\widehat{K}_t + [(1+\eta_j)\alpha_j - 1]\,\widehat{N}_t^j + \sum_{i\neq j}[(1+\eta_i)\alpha_i]\,\widehat{N}_t^i$$

thus the slope $\frac{\partial \widehat{w}^j}{\partial \widehat{N}_t^j}$ is equal to: $(1+\eta_j)\alpha_j - 1$.

[15]Clearly in this case the η_j coefficients have different values form the $\bar{\eta}_j$ of theorem 2.

[16]This amount to say that $(1+\eta_j)\alpha_j - 1 < 0$ for each labor type j. This signs are in general compatible with both conditions (14) or (15), for appropriate values of the other parameters.

economy will begin to converge to the stationary solution (the sink, implied by condition (15)) along a sequence of dampened oscillations.

If the stable set has instead to be a closed orbit, the situation is different. If for some reason the economy is positioned outside the fixed point of proposition 1, according to condition (14), the impact of capital $\frac{\partial \widehat{w}_t^i}{\partial \widehat{K}_t} = \epsilon_{i,k}^d$ must be smaller than required in the former case (i.e. that of the sink attractor). In other words, the "positive" effect of capital stock on labor demand functions $\partial \widehat{w}_t^i / \partial \widehat{K}_t$ should be sufficiently smaller: technically, it should exactly offset the effect of the labor types' cross elasticities so that the system can indefinitely endogenously fluctuate around the stationary state.

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